Revisiting the Eichten - Feinberg - Gromes $Q\bar{Q}$ Spin-Orbit Interaction

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Abstract

Invariant and covariant forms of the quark-antiquark interaction derived by the method of Eichten and Feinberg are considered. Relations between the various terms imposed by Lorentz transformation constraints, here called Gromes relations, are found to exist in neither case. Details of the Gromes relation proper are reconsidered and inconsistencies found that lead to a violation of covariance.

1 Introduction

Constituent quark models provide valuable insight into many hadronic phenomena - the mass spectrum as well as decay and transition observables. With the appearance of advanced probes such as CEBAF, where high resolution data on fine and hyperfine spectra to the order of 200 - 300 kev. are expected with complimentary data on hadronic production, their predictions are of increasing interest and in increasing use. In both sectors non-perturbative QCD effects are expected to be significant.

Theoretical input for modeling the non-perturbative regime has come from the strong coupling limit of lattice gauge theory. Wilson loop area asymptotics [1] for heavy static constituents lead directly to the popular linear confinement model. When quarks are given motion their Wilson loop and the potential interaction derived from it are then manifestly non-local, embodying the confining non-perturbative gluon dynamics. A proper accounting for these effects to $O(m^{-2})$ has been made in the well-known work of Eichten and Feinburg (E.F.), ref[2], where they enter implicitly. They do not however enter into reductions of relativistic equations that input only the linearly confining potential; there the semi-relativistic corrections are purely kinematical. Hence an increased interest in the E.F. spin dependent interaction and the simplifying relation of Gromes [3].

It would be difficult to overstate the impact these two results have had on particle physics calculations since their appearance and general acceptance over a decade ago (Spires preprint database alone, e.g., lists a combined $\approx$ 700 citation entries.); it extends from phenomenological modeling to lattice calculations. The aim of the present article is to help clarify their meanings and to set forth criticisms in the most accessible terms available. The results are reviewed separately beginning in the next section with that of ref[2] where it is demonstrated that whether in its invariant or covariant form there are with the potential no accompanying Gromes relations(G.R.). In the section that follows the G.R. proper is treated by way of its derivation and is found to violate Lorentz covariance. The violation is in con-

sequence of two main errors in the derivation: 1) discrepancy between the Lorentz transformation and the one employed there as such, and, 2) imposing of Lorentz invariance where covariance is required.

2 E.F. spin-orbit potential

Beginning from the gauge invariant quark-antiquark transition amplitude, a linearly confining potential for static quarks is defined in terms of the Wilson loop

$$G_{I,c} = \langle 0|T^* \bar{\psi}^c(y_2) P(y_2, y_1) \psi(y_1) \bar{\psi}(x_1) P(x_1, x_2) \psi^c(x_2)|0\rangle \tag{1}$$

$$\rightarrow \langle tr_D tr_\lambda \; S_0(y_1, x_1; A) P(x_1, x_2) C^{-1} S_0(x_2, y_2; A) C P(y_2, y_1)\rangle \tag{2}$$

$$= -tr_D \left(\frac{1+\gamma^0}{2} \otimes \frac{1-\gamma^0}{2} \, \tilde{I} \right) e^{-i(m_1+m_2)T} \delta(\mathbf{x}_1 - \mathbf{y}_1) \delta(\mathbf{x}_2 - \mathbf{y}_2) \,, \tag{3}$$

$$\epsilon(r) \equiv -\frac{1}{T} \ln(\tilde{I}) = -\frac{1}{T} \ln \langle tr P \exp(\imath g \oint dz_\mu A^\mu(z))\rangle \equiv -\frac{1}{T} \ln\langle 1\rangle_W \tag{4}$$

for large $T \equiv x^0 - y^0$. The symbols above have the following meanings: T^* time orders, C charge conjugates, the Ps are path-ordered exponentials, $P(x,y) = P \exp[\imath g \int_y^x dz_\mu A^\mu(z)]$, $tr_D(tr_\lambda)$ is the trace operator for Dirac (gauge) matrices, and the average is taken over the gauge fields, $\langle U\rangle \equiv \int [dA^\mu] U e^{-S_{YM}(A)}$. It is to be understood that $A_\mu \equiv A_\mu^a \lambda^a$ where A_μ^a are the QCD gauge fields and $\{\lambda^a\}$ are the representation matrices for the fermions in the fundamental representation of the gauge group SU(3). In (2) S_0 is a static fermion propagator. The analysis of ref[2] begins with the introduction of propagators for quarks in motion. They satisfy

$$(\not{D} - m)\, S(x, y; A) \;\; = \;\; \delta^4(x - y) \tag{5}$$

and may be expanded about the static propagator

$$S(x, y; A) \;\; = \;\; S_0(x, y; A) + \int d^4 z S_0(x, z; A) \boldsymbol{\gamma} \cdot \mathbf{D} S(z, y; A) \tag{6}$$

where S_0 obeys

$$(D_0 \gamma^0 - m)\, S_0(x, y; A) \;\; = \;\; \delta^4(x - y) \tag{7}$$

and is given explicitly by

$$
\begin{aligned}
S_0(x,y;A) &= -\imath\theta(x^0-y^0)e^{-\imath m(x^0-y^0)}\frac{1+\gamma^0}{2}P(x^0,y^0)\delta(\mathbf{x}-\mathbf{y}) \\
&\qquad -\imath\theta(y^0-x^0)e^{-\imath m(y^0-x^0)}\frac{1-\gamma^0}{2}P(x^0,y^0)\delta(\mathbf{x}-\mathbf{y})\,.
\end{aligned} \tag{8}
$$

To order m^{-2}, with $x^0 > y^0$, the non-static propagator is given by

$$
\begin{aligned}
&\left[1+\frac{1}{4m^2}(\mathbf{D}^2-g\boldsymbol{\sigma}\cdot\mathbf{B})\right]S^{++}(x,y;A) = S_0^{++}(x,y;A) \\
&\quad +\int d^4\omega S_0^{++}(x,\omega;A)\left[\frac{1}{2m}(\mathbf{D}^2-g\boldsymbol{\sigma}\cdot\mathbf{B})+\frac{\imath g}{4m^2}(\delta_{ij}-\imath\epsilon_{ijk}\sigma^k)E^iD^j\right]S^{++}(\omega,y;A)
\end{aligned} \tag{9}
$$

where the projections are, $S^{++}\equiv\frac{1+\gamma^0}{2}\,S\,\frac{1+\gamma^0}{2}$, $S^{+-}\equiv\frac{1+\gamma^0}{2}\,S\,\frac{1-\gamma^0}{2}$, and so forth. To obtain leading relativistic corrections to the static interaction, $\epsilon(r)$ of (4), the static fermion propagators of (2) are replaced by these nonstatic ones. Then

$$
\begin{aligned}
G_{I,c} &= \langle tr_D tr_\lambda(S_1^{++}+S_1^{+-}+S_1^{-+}+S_1^{--}) \\
&\qquad \times P(y_1,y_2)C^{-1}(S_2^{++}+S_2^{+-}+S_2^{-+}+S_2^{--})CP(x_2,x_1)\rangle \\
&\sim -tr_D\left(\frac{1+\gamma^0}{2}\otimes\frac{1-\gamma^0}{2}\tilde{I}_{l,c}+\frac{1-\gamma^0}{2}\otimes\frac{1-\gamma^0}{2}\tilde{I}_{s1}+\frac{1+\gamma^0}{2}\otimes\frac{1+\gamma^0}{2}\tilde{I}_{s2}\right) \\
&= -tr_p(\tilde{I}_{l,c}+\tilde{I}_{s1}+\tilde{I}_{s2}) = -tr_p(\tilde{I}_{l,c}+\tilde{I}_{s,c})\equiv -tr_p\,\tilde{G}_{p,c}
\end{aligned} \tag{10, 11}
$$

where $I_{l,c}(I_{s,c})$ is identified as the antiquark-charge-conjugated large(small) 2x2 Pauli component and tr_p is the trace operator for Pauli matrices. When the large component only is accounted for the combined static and $O(m^{-2})$ spin-orbit corrections are

$$
V = -\frac{1}{T}(\tilde{I}_{l,c}) = \epsilon(r) \tag{12}
$$

$$
+\left(\frac{\mathbf{s}_1\cdot\mathbf{L}_1}{2m_1^2}-\frac{\mathbf{s}_2\cdot\mathbf{L}_2}{2m_2^2}\right)\frac{\epsilon'(r)}{r} \tag{13}
$$

$$
+\left(\frac{\mathbf{s}_1\cdot\mathbf{L}_1}{m_1^2}-\frac{\mathbf{s}_2\cdot\mathbf{L}_2}{m_2^2}\right)\frac{V_1'}{r}+\left(\frac{\mathbf{s}_2\cdot\mathbf{L}_1}{m_1m_2}-\frac{\mathbf{s}_1\cdot\mathbf{L}_2}{m_1m_2}\right)\frac{V_2'}{r} \tag{14}
$$

$$
-\left(\frac{\mathbf{s}_1}{m_1}\cdot\frac{1}{T}\int_{-T/2}^{T/2}dz\langle\imath g\mathbf{B}(\mathbf{x}_1,z)\rangle_W/\langle 1\rangle_W-(1\to 2)\right) \tag{15}
$$

where, $\mathbf{L}_i \equiv \mathbf{r} \times \mathbf{p}_i$, and

$$\left(\frac{\mathbf{s}_1 \cdot \mathbf{L}_1}{m_1^2} - \frac{\mathbf{s}_2 \cdot \mathbf{L}_2}{m_2^2}\right)\frac{V_1'}{r} \equiv -\Bigg(\frac{\mathbf{s}_1}{2m_1^2} \cdot \frac{1}{T}\int\int_{-T/2}^{T/2} dzdz' \langle \mathbf{B}(\mathbf{x}_1, z)\mathbf{D}^2(\mathbf{x}_1, z')\rangle_W / \langle 1 \rangle_W \\ -(1 \rightarrow 2)\Bigg) \tag{16}$$

$$(\mathbf{s}_2 \cdot \mathbf{L}_1 - \mathbf{s}_1 \cdot \mathbf{L}_2)\frac{V_2'}{r} \equiv -\Bigg(\frac{\mathbf{s}_1}{2} \cdot \frac{1}{T}\int\int_{-T/2}^{T/2} dzdz' \langle \mathbf{B}(\mathbf{x}_1, z)\mathbf{D}^2(\mathbf{x}_2, z')\rangle_W / \langle 1 \rangle_W \\ -(1 \leftrightarrow 2)\Bigg). \tag{17}$$

A few remarks concerning this result are in order. First, the beginning four point function (1) differs from that of ref [2] where the antifermion fields have not been charge conjugated. This difference in four-point functions naturally yields a corresponding difference in the interactions derived from them. Second, in the interaction potential derivation of [2] there are several algebraic errors leading to the final result there (see appendix for details). The interaction resulting from the algebraically correct non-charge-conjugated derivation we identify in this paper as the E.F. result, V_{EF} of the appendix. This E.F. potential contrasts with the above potential most strikingly in that quark and antiquark contributions (for each term type) appear in the E.F. potential with identical algebraic sign. In the potential above however, derived in the appendix as $V_{EF,c}$, they appear with opposite sign as one would expect for a $q\bar{q}$ state. For this reason the four-point function of (1) is seen to have a more acceptable interpretation as a $q\bar{q}$ propagator than the beginning non-charge-conjugated four-point function of [2] leading to the E.F. result.

In ref[3] it is pointed out that insofar as the derived interaction is related to a v.e.v. in a Lorentz invariant theory the interaction itself must also be invariant. In fact, V as given in (12), is not Lorentz invariant without a G.R.. It could be argued that omission of the small component $\tilde{I}_{s,c}$ in the potential definition (12) has ruined the invariance since $\tilde{I}_{s,c}$ appears in the invariant 4-point function (11) on an equal footing with the large component. That this is not the case is shown by simply replacing $\tilde{I}_{l,c}$ in (12) with $\tilde{G}_{p,c}$ of (11) and then testing the resulting

potential for invariance. The addition to V on this replacement exactly cancels line (13), the classical spin-orbit and Thomas precession line, resulting in a spin-orbit interaction which we denote $V_{I,c}$. Details of the derivation are given in the appendix, equations (77)-(101). We now consider a Lorentz boost of $V_{I,c}$ to leading order in velocity. Care should be taken when transforming the static propagators since by construction they must remain solutions to the non-covariant equation (7). To leading order their transformation is

$$S_0^{\pm\pm}(x,y;A) \;\;\rightarrow\;\; S_0^{\pm\pm}(x,y;A) \pm \int d^4z S_0^{\pm\pm}(x,z;A)\mathbf{v}\cdot\mathbf{D}(\mathbf{x},z)S_0^{\pm\pm}(z,y;A) \tag{18}$$

which is effectively carried out on their time-like path ordered exponentials as

$$P(x^0,y^0) \;\;\rightarrow\;\; P(x^0,y^0) - \imath\int_{y^0}^{x^0} dz P(x^0,z)\mathbf{v}\cdot\mathbf{D}(\mathbf{x},z)P(z,y^0)\,. \tag{19}$$

And so the effective transformation (19), carried out in (15), cancels with the momentum transformation of (14), and the remaining magnetic field transformation of (15) yields $V_{I,c} \rightarrow V_{I,c} + (\mathbf{s}_1\cdot[\mathbf{r}\times\mathbf{v}]/m_1 - \mathbf{s}_2\cdot[\mathbf{r}\times\mathbf{v}]/m_2)\epsilon'(r)/r$, and noninvariance.

What must be remembered about the v.e.v. $G_{I,c}$ of (10) is that its Lorentz invariance depends crucially upon its Dirac (or equivalently, its Pauli) trace operator without which it is not invariant

$$G_{I,c} \;\;\rightarrow\;\; G_{I,c} + \delta G \tag{20}$$

$$\delta G \;\;\sim\;\; tr_p\Big(\mathbf{1}_{2x2}\otimes\frac{\boldsymbol{\sigma}}{m_2}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} dz\langle\mathbf{E}(\mathbf{x}_2,z)\rangle_W\times\mathbf{v} \tag{21}$$

$$+\frac{\boldsymbol{\sigma}}{m_1}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} dz\langle\mathbf{E}(\mathbf{x}_1,z)\rangle_W\times\mathbf{v}\otimes\mathbf{1}_{2x2}\Big) \rightarrow 0 \tag{22}$$

(see appendix for details). From the potential definition (12) both as given or with $\tilde{G}_{p,c}$ in place of the large component $\tilde{I}_{l,c}$ it is clear that Lorentz invariance of this potential related as it is to $G_{I,c}$ is not to be expected, cannot be argued as it stands, i.e., without the operation of the Pauli trace on $\tilde{I}_{l,c}$ (or $\tilde{G}_{p,c}$). In such an operation of course any spin dependence is averaged over, destroyed. Hence there remains

no argument for the Lorentz invariance of the spin-dependent contribution to $V_{I,c}$ derived as it is from the invariant $G_{I,c}$, and so no argument for Gromes relations concerning $V_{I,c}$.

While $V_{I,c}$ like $V_{EF,c}$ is not Lorentz invariant, it does account consistently for the $O(m^{-2})$ contributions to the invariant 4-point function from which it is derived, while $V_{EF,c}$ does not. Considered as a $Q\bar{Q}$ interaction Hamiltonian however $V_{I,c}$ has consistency problems of its own. A simple example will suffice to make the point. It is well-known that in the heavy antiquark limit, $m_2 \to \infty$, lines (12) and (13) agree exactly with the standard m_1^{-2} expansion of the Dirac equation in Pauli form in the presence of a central electric field. Then with the absence of line (13) in $V_{I,c}$ there is disagreement between $V_{I,c}$ and the standard reduction of the Dirac equation.

There is no such disagreement with the interaction derived from a four-point function whose Lorentz spinor structure is maintained (see appendix for details). This potential we denote V_{cov} since it is derived from the covariant four-point function. Reduction of the covariant four-point function's off diagonal elements to $O(m^{-2})$ is performed by the Foldy-Wouthysen transformation, $U = \exp(\imath s(\xi))$, where, $s(\xi) = \imath\gamma^0\boldsymbol{\gamma}\cdot\mathbf{D}(\xi)/2m$ with a resulting interaction equivalent to V of (12). I.e., the covariant interaction is equivalent to the invariant interaction when the small component contribution to the invariant trace is (improperly) omitted. Its Lorentz transformation properties are unambiguous. From the lowest order transformation, $\mathbf{p}_i \to \mathbf{p}_i - m_i\mathbf{v}$, $\mathbf{B} \to \mathbf{B} - \mathbf{v}\times\mathbf{E}$, and (19), V transforms as, $V \to V + (\mathbf{s}_1\cdot[\mathbf{r}\times\mathbf{v}]/2m_1 - \mathbf{s}_2\cdot[\mathbf{r}\times\mathbf{v}]/2m_2)\epsilon'(r)/r \equiv V + \tilde{V}$ (notice that neither V_1 nor V_2 contribute to $\tilde{V}$) in strict agreement with its covariant transformation as a product of spinor products

$$V \sim S_1^{++}\otimes C^{-1}S_2^{--}C \;\to\; LS_1^{++}L^{-1}\otimes LC^{-1}S_2^{--}CL^{-1} \tag{23}$$

$$\approx (S_1^{++} - \frac{1}{2}\mathbf{v}\cdot\boldsymbol{\gamma}S_1^{-+} - \frac{1}{2}S_1^{+-}\mathbf{v}\cdot\boldsymbol{\gamma})\otimes \tag{24}$$

$$C^{-1}(S_2^{--} + \frac{1}{2}S_2^{-+}\mathbf{v}\cdot\boldsymbol{\gamma} + \frac{1}{2}\mathbf{v}\cdot\boldsymbol{\gamma}S_2^{+-})C \tag{25}$$

$$\sim \quad V + \tilde{V} \tag{26}$$

where $L \approx 1 - \boldsymbol{\alpha} \cdot \mathbf{v}/2$, again, as in the invariant case, precluding the existence of Lorentz constraining relations between the transformed terms.

3 The Gromes method of derivation

In view of the previous discussions a detailed examination of the G.R. still seems worthwhile if only to answer to the enormous interest it has generated over the last decade and a half. It should be clear however that such an examination is little more than an academic exercise. In ref[3] the lowest order Lorentz transformation on V by which the relation is derived is implemented in three distinct steps.

1)momentum transformation:

$$\mathbf{p}_i \to \mathbf{p}_i - m_i \mathbf{v}\,.$$

2)magnetic field transformation:

$$\mathbf{B} \to \mathbf{B} - \mathbf{v} \times \mathbf{E}$$

which by the given lemma, (2.8) of [3], is effectively carried out on the relative coordinate as

$$\mathbf{r} \to \mathbf{r} + \mathbf{v} \times \mathbf{s}_1/m_1 - \mathbf{v} \times \mathbf{s}_2/m_2\,.$$

3) $\mathbf{r}_i$ transformation:

$$\mathbf{r}_i \to \mathbf{r}_i + [\mathbf{v} \times \mathbf{s}_i]/2m_i$$

following the definition and transformation specification of the quantity

$$\begin{aligned}\mathbf{x} &\equiv \mathbf{r}_i + (\mathbf{p}_i/m_i)t + [\mathbf{p}_i \times \mathbf{s}_i]/2m_i \\ \mathbf{x} &\to \mathbf{x} - \mathbf{v}t\,.\end{aligned}$$

Effects of transformations 1) and 2) taken together with the effective transformation of the path-ordered exponentials, (19), combine in V of (12) to give the correct covariant transformation (23), $V \to V + \tilde{V}$.

Transformation 3) however is entirely spurious with no correspondence to the field theoretic Lorentz transformation. Its effect is to produce the extra transformation terms, $\epsilon(r) \to \epsilon(r) + ([\mathbf{v} \times \mathbf{s}_1] \cdot \mathbf{r}/2m_1 - [\mathbf{v} \times \mathbf{s}_2] \cdot \mathbf{r}/2m_2)\epsilon'(r)/r$. The G.R. appears when 1), 2), and 3) are taken together (with a sign error included for the terms from step 2)) and invariance imposed to $O(m^{-1})$. Then, $\epsilon + V_1 - V_2 = 0$.

The relation has received derivation in several other contexts as well. These derivations should each one undergo equal scrutiny.

4 Summary

The E.F. interaction Hamiltonian contains dynamical information on the nonperturbative gluon field not present in models that simply relativize the static linear potential. But in its given form one is left either to make simplifying assumptions or to pursue evaluation on the lattice. For the latter, it is unclear whether the sign discrepancies between the results here, (14) - (16), and (V_1, V_2) of ref[2] would affect corresponding discrepancies between the respective lattice reductions. For the former, the electric confinement ansatz, $V_1 = V_2 = 0$, in conceptual agreement with Buchmüller's picture[4], assumed in [2] and later abandoned (due to variance with spin phenomenology), demonstrates that these should be made only with special care and that they may in any event have consequences difficult to predict. In these regards it may be viewed as a shortcoming of the E.F. formalism that here too only the static limit of the minimal area law enters explicitly; nonlocality is implied only (e.g., via electric field insertions [3]). This happens as an artifact of the initial fermion propagator expansion (6) around the static limit, and is readily remedied when these propagators are replaced by ones more compatible with semi-relativistic

fermion motion. Then the Wilson loop of (4) is explicitly nonlocal and relates the resulting potential directly to the minimum area.

Such a program has been carried through in an article by Brambilla, Consoli, and Prosperi [5] both for spin and spin-independent corrections to $O(m^{-3})$. From propagators expressed as integrals over the phase space and the Nambu-Goto action as the effective area their final potential is given in terms of familiar quantum mechanical operators (It should be pointed out that the spin-orbit agreement with ref[3] follows from a systematic mathematical error made in their appendix [6]. See also ref[7].).

A more rigorous implementation of the electric confinement ansatz might also be attempted, though it is uncertain whether Buchmüller's picture remains self-consistent for quarks in motion or whether it agrees with minimal area asymptotics. These questions are considered in reference [7].

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5 appendix: potential derivations

In evaluating the gauge-invariant $Q\bar{Q}$ 4-point function

$$G = \langle 0|T^* \bar{\psi}(y_2)P(y_2,y_1)\psi(y_1)\bar{\psi}(x_1)P(x_1,x_2)\psi(x_2)|0\rangle \tag{27}$$

$$\rightarrow \langle trS(x_2,y_2;A)P(y_2,y_1)S(y_1,x_1;A)P(x_1,x_2)\rangle \tag{28}$$

the trace appearing in (28) may be taken over both gauge and Dirac matrices or over gauge matrices only, depending, respectively, on whether the four suppressed Dirac indices of (27) are of two like pairs or are all distinct. In the first instance the

four point function is invariant under Lorentz transformations and in the second it is covariant, transforming as a product of spinor products. Interaction potentials derived from these invariant and covariant 4-point functions therefore have transformation properties that are invariant and covariant, respectively. Both 4-point functions appear in the literature as starting points for the potential derivation; for example, in [2] the invariant version is used, and in [5] the covariant version.

Interaction potentials from the invariant 4-point function :

First we derive interaction potentials from the invariant 4-point function. The derivations follow closely that given in [2]. Fermion propagators of (28) are expanded around the static propagator as

$$S(x,y;A) = S_0(x,y;A) + \int d^4z S_0(x,z;A)\boldsymbol{\gamma}\cdot\mathbf{D}S(z,y;A) \tag{29}$$

with

$$\begin{aligned} S_0(x,y;A) &= -\imath\theta(x^0-y^0)e^{-\imath m(x^0-y^0)}\frac{1+\gamma^0}{2}P(x^0,y^0)\delta(\mathbf{x}-\mathbf{y}) \\ &\quad -\imath\theta(y^0-x^0)e^{-\imath m(y^0-x^0)}\frac{1-\gamma^0}{2}P(x^0,y^0)\delta(\mathbf{x}-\mathbf{y})\,. \end{aligned} \tag{30}$$

They are then expanded over static energy projectors

$$\begin{aligned} S &= \frac{1+\gamma^0}{2}S\frac{1+\gamma^0}{2} + \frac{1+\gamma^0}{2}S\frac{1-\gamma^0}{2} + \frac{1-\gamma^0}{2}S\frac{1+\gamma^0}{2} + \frac{1-\gamma^0}{2}S\frac{1-\gamma^0}{2} \\ &\equiv S^{++} + S^{+-} + S^{-+} + S^{--} \end{aligned} \tag{31}$$

so that

$$S^{++}(x,y;A) = S_0^{++}(x,y;A) + \int d^4z S_0^{++}(x,z;A)\boldsymbol{\gamma}\cdot\mathbf{D}(z)S^{-+}(z,y;A) \tag{32}$$

$$S^{+-}(x,y;A) = \int d^4z S_0^{++}(x,z;A)\boldsymbol{\gamma}\cdot\mathbf{D}(z)S^{--}(z,y;A) \tag{33}$$

$$S^{-+}(x,y;A) = \int d^4z S_0^{--}(x,z;A)\boldsymbol{\gamma}\cdot\mathbf{D}(z)S^{++}(z,y;A) \tag{34}$$

$$S^{--}(x,y;A) = S_0^{--}(x,y;A) + \int d^4z S_0^{--}(x,z;A)\boldsymbol{\gamma}\cdot\mathbf{D}(z)S^{+-}(z,y;A) \tag{35}$$

The projected propagator components above are evaluated to $O(m^{-2})$ by substitution of the static propagator solution (30) and iteration to this order for $x^0 > y^0$. For the S^{++} component then

$$S^{++}(x,y) \;=\; S_0^{++}(x,y) + \int d^4w \Delta^{+-}(x,w)\boldsymbol{\gamma}\cdot\mathbf{D}(w)S^{++}(w,y) \tag{36}$$

where

$$\begin{aligned}
\Delta^{+-}(x,w) &= \int d^4z S_0^{++}(x,z)\boldsymbol{\gamma}\cdot\mathbf{D}(z)S_0^{--}(z,w) \\
&= -\frac{1+\gamma^0}{2}e^{-im(x^0+w^0)}\delta(\mathbf{x}-\mathbf{w})\int dz_0\theta(x^0-z^0)\theta(w^0-z^0)e^{2imz^0}f(z^0) \\
&\equiv -\frac{1+\gamma^0}{2}e^{-im(x^0+w^0)}\delta(\mathbf{x}-\mathbf{w})I
\end{aligned} \tag{37}$$

and

$$f(z^0) \;\equiv\; P(x^0,z^0)\boldsymbol{\gamma}\cdot\mathbf{D}(z)P(z^0,w^0)\,. \tag{38}$$

We rewrite the above integral as

$$I \;=\; e^{2imw^0}\theta(x^0-w^0)\int_{-\infty}^{0} dz^0 e^{2imz^0}f(z^0+w^0) \tag{39}$$

$$+e^{2imx^0}\theta(w^0-x^0)\int_{-\infty}^{0} dz^0 e^{2imz^0}f(z^0+x^0)\,. \tag{40}$$

and use the $O(m^{-2})$ integral relation

$$\int_{-\infty}^{0} dz^0 e^{2imz^0}f(z^0+\xi) \;\simeq\; -\frac{i}{2m}f(\xi)+\frac{1}{4m^2}f'(\xi) \tag{41}$$

with

$$f(w^0) \;=\; P(x^0,w^0)\boldsymbol{\gamma}\cdot\mathbf{D}(w)\,, \quad f'(w^0) = -iP(x^0,w^0)[D^0(w),\boldsymbol{\gamma}\cdot\mathbf{D}(w)] \tag{42}$$

$$f(x^0) \;=\; \boldsymbol{\gamma}\cdot\mathbf{D}(x)P(x^0,w^0)\,, \quad f'(x^0) = -i[D^0(x),\boldsymbol{\gamma}\cdot\mathbf{D}(x)]P(x^0,w^0) \tag{43}$$

to find on substitution

$$\Delta^{+-}(x,w) \;=\; -S_0^{++}(x,w)(\frac{1}{2m}\boldsymbol{\gamma}\cdot\mathbf{D}(w)+\frac{1}{4m^2}[D^0(w),\boldsymbol{\gamma}\cdot\mathbf{D}(w)]) \tag{44}$$

$$-(\frac{1}{2m}\boldsymbol{\gamma}\cdot\mathbf{D}(x)+\frac{1}{4m^2}[D^0(x),\boldsymbol{\gamma}\cdot\mathbf{D}(x)])S_0^{--}(x,w) \tag{45}$$

and

$$
\begin{aligned}
S^{++}(x,y) &= S_0^{++}(x,y) - \int d^4w S_0^{++}(x,w)(\frac{1}{2m}\boldsymbol{\gamma}\cdot\mathbf{D}(w) && (46)\\
&\quad + \frac{1}{4m^2}[D^0(w),\boldsymbol{\gamma}\cdot\mathbf{D}(w)])\boldsymbol{\gamma}\cdot\mathbf{D}(w)S^{++}(w,y) && (47)\\
&\quad -(\frac{1}{2m}\boldsymbol{\gamma}\cdot\mathbf{D}(x) + \frac{1}{4m^2}[D^0(x),\boldsymbol{\gamma}\cdot\mathbf{D}(x)])\tilde{\Delta}^{-+}(x,y) && (48)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\Delta}^{-+}(x,y) &= \int d^4w S_0^{--}(x,w)\boldsymbol{\gamma}\cdot\mathbf{D}(w)S^{++}(w,y) && (49)\\
&\simeq -\frac{1}{2m}\boldsymbol{\gamma}\cdot\mathbf{D}(x)S^{++}(x,y)\,. && (50)
\end{aligned}
$$

With this, and from the identities

$$
\begin{aligned}
(\boldsymbol{\gamma}\cdot\mathbf{D})^2 &= -\mathbf{D}^2 + g\boldsymbol{\sigma}\cdot\mathbf{B} && (51)\\
\left[D^0,\boldsymbol{\gamma}\cdot\mathbf{D}\right]\boldsymbol{\gamma}\cdot\mathbf{D} &= -ig(\delta_{ij} - \imath\epsilon_{ijk}\sigma^k)E^iD^j && (52)
\end{aligned}
$$

we arrive at

$$
\begin{aligned}
&\left[1 + \frac{1}{4m^2}(\mathbf{D}^2 - g\boldsymbol{\sigma}\cdot\mathbf{B})\right]S^{++}(x,y) = S_0^{++}(x,y) && (53)\\
&\quad + \int d^4\omega S_0^{++}(x,\omega)\left[\frac{1}{2m}(\mathbf{D}^2 - g\boldsymbol{\sigma}\cdot\mathbf{B}) + \frac{\imath g}{4m^2}(\delta_{ij} - \imath\epsilon_{ijk}\sigma^k)E^iD^j\right]S^{++}(\omega,y)
\end{aligned}
$$

equation (9). Then finally, to $O(m^{-2})$

$$
\begin{aligned}
S^{++}(x,y) &\simeq S_0^{++}(x,y) + \int d^4\omega S_0^{++}(x,\omega)\Big[\frac{1}{2m}(\mathbf{D}^2 - g\boldsymbol{\sigma}\cdot\mathbf{B}) && (54)\\
&\qquad + \frac{\imath g}{4m^2}(\delta_{ij} - \imath\epsilon_{ijk}\sigma^k)E^iD^j\Big]S_0^{++}(\omega,y) && (55)\\
&\quad + \frac{1}{4m^2}\int d^4\omega d^4z S_0^{++}(x,\omega)(\mathbf{D}^2 - g\boldsymbol{\sigma}\cdot\mathbf{B}) && (56)\\
&\qquad \times S_0^{++}(\omega,z)(\mathbf{D}^2 - g\boldsymbol{\sigma}\cdot\mathbf{B})S_0^{++}(z,y) && (57)\\
&\quad - \frac{1}{4m^2}(\mathbf{D}^2 - g\boldsymbol{\sigma}\cdot\mathbf{B})S_0^{++}(x,y)\,. && (58)
\end{aligned}
$$

By similar methods we find ($x^0 > y^0$)

$$S^{+-}(x,y) \simeq -\frac{1}{2m} S_0^{++}(x,y) \boldsymbol{\gamma}\cdot\mathbf{D}(y) \tag{59}$$

$$S^{-+}(x,y) \simeq -\frac{1}{2m} \boldsymbol{\gamma}\cdot\mathbf{D}(x) S_0^{++}(x,y) \tag{60}$$

$$S^{--}(x,y) \simeq \frac{1}{4m^2} \boldsymbol{\gamma}\cdot\mathbf{D}(x) S_0^{++}(x,y) \boldsymbol{\gamma}\cdot\mathbf{D}(y) \tag{61}$$

$$= -\frac{1}{4m^2}\Big[\tilde{S}_0^{--}(x,y)[\mathbf{D}^2(y) - g\boldsymbol{\sigma}\cdot\mathbf{B}(y)] \tag{62}$$

$$+ig\int d^4z \tilde{S}_0^{--}(x,z)(\delta_{ij} - \imath\epsilon_{ijk}\sigma^k)E^i(z)\tilde{S}_0^{--}(z,y)D^j(y)\Big] \tag{63}$$

and

$$S^{--}(y,x) \simeq S_0^{--}(y,x) + \int d^4\omega S_0^{--}(y,\omega)\Big[\frac{1}{2m}(\mathbf{D}^2 - g\boldsymbol{\sigma}\cdot\mathbf{B}) \tag{64}$$

$$-\frac{\imath g}{4m^2}(\delta_{ij} - \imath\epsilon_{ijk}\sigma^k)E^i D^j\Big] S_0^{--}(\omega,x) \tag{65}$$

$$+\frac{1}{4m^2}\int d^4\omega d^4z S_0^{--}(y,\omega)(\mathbf{D}^2 - g\boldsymbol{\sigma}\cdot\mathbf{B}) \tag{66}$$

$$\times S_0^{--}(\omega,z)(\mathbf{D}^2 - g\boldsymbol{\sigma}\cdot\mathbf{B}) S_0^{--}(z,x) \tag{67}$$

$$-\frac{1}{4m^2}(\mathbf{D}^2 - g\boldsymbol{\sigma}\cdot\mathbf{B}) S_0^{--}(y,x) \tag{68}$$

$$S^{-+}(y,x) \simeq -\frac{1}{2m} S_0^{--}(y,x) \boldsymbol{\gamma}\cdot\mathbf{D}(x) \tag{69}$$

$$S^{+-}(y,x) \simeq -\frac{1}{2m} \boldsymbol{\gamma}\cdot\mathbf{D}(y) S_0^{--}(y,x) \tag{70}$$

$$S^{++}(y,x) \simeq \frac{1}{4m^2} \boldsymbol{\gamma}\cdot\mathbf{D}(y) S_0^{--}(y,x) \boldsymbol{\gamma}\cdot\mathbf{D}(x) \tag{71}$$

$$= -\frac{1}{4m^2}\Big[\tilde{S}_0^{++}(y,x)[\mathbf{D}^2(x) - g\boldsymbol{\sigma}\cdot\mathbf{B}(x)] \tag{72}$$

$$-ig\int d^4z \tilde{S}_0^{++}(y,z)(\delta_{ij} - \imath\epsilon_{ijk}\sigma^k)E^i(z)\tilde{S}_0^{++}(z,x)D^j(x)\Big] \tag{73}$$

where the identity

$$\boldsymbol{\gamma}\cdot\mathbf{D}(x) S_0^{\pm\pm}(x,y) = \tilde{S}_0^{\mp\mp}(x,y)\boldsymbol{\gamma}\cdot\mathbf{D}(y) \pm i\int d^4z \tilde{S}_0^{\mp\mp}(x,z)\boldsymbol{\gamma}\cdot\mathbf{E}(z) S_0^{\pm\pm}(z,y) \tag{74}$$

with

$$\tilde{S}_0(x,y;A) \equiv -\imath\theta(x^0-y^0)e^{-\imath m(x^0-y^0)}\frac{1-\gamma^0}{2}P(x^0,y^0)\delta(\mathbf{x}-\mathbf{y}) -\imath\theta(y^0-x^0)e^{-\imath m(y^0-x^0)}\frac{1+\gamma^0}{2}P(x^0,y^0)\delta(\mathbf{x}-\mathbf{y}) \quad (75)$$

has been used, and the off-diagonals, S^{+-} and S^{-+} are shown to lowest order. Note that in evaluating $S^{--}(y,x)$ above the integral relation used in place of equation (41) (which is used in the evaluation of $S^{++}(x,y)$) is

$$\int_0^\infty dz^0 e^{-2imz^0} f(z^0+\xi) \simeq -\frac{i}{2m}f(\xi) - \frac{1}{4m^2}f'(\xi) \quad (76)$$

yielding the algebraic sign difference between lines (55) and (65). For construction of the potential interaction we express the invariant Green's function as

$$G_I = \langle tr\, S_1\, P\, S_2\, P\rangle \quad (77)$$

$$= tr_D\langle tr_\lambda\, S_1\, P\, S_2\, P\rangle \quad (78)$$

$$\equiv -tr_D\tilde{G}e^{-i(m_1+m_2)(x^0-y^0)}\delta(\mathbf{x}_1-\mathbf{y}_1)\delta(\mathbf{x}_2-\mathbf{y}_2) \quad (79)$$

$$= -tr_p\tilde{G}_p e^{-i(m_1+m_2)(x^0-y^0)}\delta(\mathbf{x}_1-\mathbf{y}_1)\delta(\mathbf{x}_2-\mathbf{y}_2) \quad (80)$$

where in (78) Dirac and gauge traces have been explicitly separated, tr_p is the trace operator for Pauli matrices, and $\tilde{G}_p$ is the sum of 2x2 Pauli components along the diagonal of $\tilde{G}$. The "invariant" potential interaction is then given by

$$V_I = -\frac{1}{T}\ln\tilde{G}_p\,. \quad (81)$$

for large T. Substitution of (54) - (73) into (28) then yields for the invariant 4-point function

$$G_I = \langle tr_D tr_\lambda[S^{++}(x_1,y_1)+S^{--}(x_1,y_1)]P(y_1,y_2) \quad (82)$$

$$\times[S^{--}(y_2,x_2)+S^{++}(y_2,x_2)]P(x_2,x_1)\rangle \quad (83)$$

$$\equiv -tr_D\langle tr_\lambda P\left[\frac{1+\gamma^0}{2}S^+_{1p}+\frac{1-\gamma^0}{2}S^-_{1p}\right]\otimes\left[\frac{1-\gamma^0}{2}S^-_{2p}+\frac{1+\gamma^0}{2}S^+_{2p}\right] \quad (84)$$

$$\times\exp(\imath g\oint dz_\mu A^\mu(z))\rangle e^{-i(m_1+m_2)T}\delta(\mathbf{x}_1-\mathbf{y}_1)\delta(\mathbf{x}_2-\mathbf{y}_2) \quad (85)$$

where $x^0 = -y^0 = \frac{T}{2}$ has been taken, and the "p" subscript identifies Pauli components which are given by

$$
\begin{aligned}
S^+_{1p} &= 1 - \frac{i}{2m_1}\int_{-\frac{T}{2}}^{\frac{T}{2}} dz(\mathbf{D}^2(\mathbf{x}_1,z) - \boldsymbol{\sigma}_1\cdot\mathbf{B}(\mathbf{x}_1,z)) && (86)\\
&\quad -\frac{i}{4m_1^2}\int_{-\frac{T}{2}}^{\frac{T}{2}} dz\ \epsilon_{ijk}\sigma_1^k E^i(\mathbf{x}_1,z)D^j(\mathbf{x}_1,z) && (87)\\
S^-_{1p} &= \frac{i}{4m_1^2}\int_{-\frac{T}{2}}^{\frac{T}{2}} dz\ \epsilon_{ijk}\sigma_1^k E^i(\mathbf{x}_1,z)D^j(\mathbf{x}_1,-T/2) && (88)\\
S^-_{2p} &= 1 - \frac{i}{2m_2}\int_{-\frac{T}{2}}^{\frac{T}{2}} dz(\mathbf{D}^2(\mathbf{x}_2,z) - \boldsymbol{\sigma}_2\cdot\mathbf{B}(\mathbf{x}_2,z)) && (89)\\
&\quad +\frac{i}{4m_2^2}\int_{-\frac{T}{2}}^{\frac{T}{2}} dz\, \epsilon_{ijk}\sigma_1^k E^i(\mathbf{x}_2,z)D^j(\mathbf{x}_2,z) && (90)\\
S^+_{2p} &= -\frac{i}{4m_2^2}\int_{-\frac{T}{2}}^{\frac{T}{2}} dz\, \epsilon_{ijk}\sigma_2^k E^i(\mathbf{x}_2,z)D^j(\mathbf{x}_2,T/2) && (91)
\end{aligned}
$$

for $O(m^{-2})$ static and spin-orbit contributions only. Gauge fields evaluated at $\pm\frac{T}{2}$ have been set to zero.

To show explicitly that the Lorentz invariance of G_I depends upon the operation of its Pauli trace we consider the leading order boost $L \simeq 1 - \boldsymbol{\alpha}\cdot\mathbf{v}/2$ on G_I

$$
\begin{aligned}
G_I \;\rightarrow\;& G_I + \delta G_I = \langle tr\, S_1\, P\, S_2\, P\rangle && (92)\\
&+\frac{1}{2}\langle tr\Big[S_0^{++}\otimes P\left([S^{-+},\mathbf{v}\cdot\boldsymbol{\gamma}] - [S^{+-},\mathbf{v}\cdot\boldsymbol{\gamma}]\right)P + && (93)\\
&\qquad P\left([S^{-+},\mathbf{v}\cdot\boldsymbol{\gamma}] - [S^{+-},\mathbf{v}\cdot\boldsymbol{\gamma}]\right)P\otimes S_0^{--}\Big]\rangle && (94)
\end{aligned}
$$

giving on substitution for $S^{\pm\mp}$ and the use of identity (74)

$$
\begin{aligned}
\delta G \;\sim\;& tr_p\Big(\mathbf{1}_{2x2}\otimes\frac{\boldsymbol{\sigma}}{m_2}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} dz\langle\mathbf{E}(\mathbf{x}_2,z)\rangle_W\times\mathbf{v} && (95)\\
&-\frac{\boldsymbol{\sigma}}{m_1}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} dz\langle\mathbf{E}(\mathbf{x}_1,z)\rangle_W\times\mathbf{v}\otimes\mathbf{1}_{2x2}\Big)\rightarrow 0\,. && (96)
\end{aligned}
$$

From (80) then

$$
\tilde{G}_p = \langle(S^+_{1p}+S^-_{1p})\otimes(S^-_{2p}+S^+_{2p})W\rangle \qquad (97)
$$

$$\simeq \tilde{I}_l + \tilde{I}_s \tag{98}$$

with

$$\tilde{I}_l \equiv \langle S_{1p}^{+} \otimes S_{2p}^{-} W \rangle \tag{99}$$

$$\tilde{I}_s \equiv \langle (S_{1p}^{-} + S_{2p}^{+}) W \rangle \tag{100}$$

where $\tilde{I}_l$ is the large component(identified by its nonzero value in the static limit, $m \to \infty$) and $W = \exp(\imath g \oint dz_\mu A^\mu(z))$ is the Wilson loop. From (81) we have for the interaction

$$\begin{aligned} V_I &= -\frac{1}{T}\ln(\tilde{I}_l + \tilde{I}_s) && (101)\\ &\simeq \epsilon(r) && (102)\\ &\quad + \left(\frac{\mathbf{s}_1 \cdot \mathbf{L}_1}{m_1^2} - \frac{\mathbf{s}_2 \cdot \mathbf{L}_2}{m_2^2}\right)\frac{V_1'}{r} + \left(\frac{\mathbf{s}_2 \cdot \mathbf{L}_1}{m_1 m_2} - \frac{\mathbf{s}_1 \cdot \mathbf{L}_2}{m_1 m_2}\right)\frac{V_2'}{r} && (103)\\ &\quad - \left(\frac{\mathbf{s}_1}{m_1} \cdot \frac{1}{T}\int_{-T/2}^{T/2} dz \langle \imath g \mathbf{B}(\mathbf{x}_1, z)\rangle_W / \langle 1 \rangle_W + (1 \to 2)\right) && (104) \end{aligned}$$

where, $\mathbf{L}_i \equiv \mathbf{r} \times \mathbf{p}_i$, and

$$\left(\frac{\mathbf{s}_1 \cdot \mathbf{L}_1}{m_1^2} - \frac{\mathbf{s}_2 \cdot \mathbf{L}_2}{m_2^2}\right)\frac{V_1'}{r} \equiv -\Bigg(\frac{\mathbf{s}_1}{2m_1^2} \cdot \frac{1}{T}\int\int_{-T/2}^{T/2} dz dz' \langle \mathbf{B}(\mathbf{x}_1, z)\mathbf{D}^2(\mathbf{x}_1, z')\rangle_W / \langle 1 \rangle_W + (1 \to 2)\Bigg) \tag{105}$$

$$(\mathbf{s}_2 \cdot \mathbf{L}_1 - \mathbf{s}_1 \cdot \mathbf{L}_2)\frac{V_2'}{r} \equiv -\Bigg(\frac{\mathbf{s}_1}{2} \cdot \frac{1}{T}\int\int_{-T/2}^{T/2} dz dz' \langle \mathbf{B}(\mathbf{x}_1, z)\mathbf{D}^2(\mathbf{x}_2, z')\rangle_W / \langle 1 \rangle_W + (1 \leftrightarrow 2)\Bigg) \tag{106}$$

and the relation

$$\imath \int_{-\frac{T}{2}}^{\frac{T}{2}} dz\, \epsilon_{ijk}\sigma^k \langle E^i(\mathbf{x}, z) D^j(\mathbf{x}, \xi)\rangle \to -Te^{-\epsilon(r)T}\, \boldsymbol{\sigma} \cdot \mathbf{L}\, \frac{\epsilon'(r)}{r} \tag{107}$$

has been used. The sum $\tilde{I}_l + \tilde{I}_s$ appearing above in (98) is to be compared with $\tilde{I}$ in equation (4.3) of [2]. When the “small” component of the invariant trace, $\tilde{I}_s$, is

ignored V_I becomes

$$\begin{aligned} V_{EF} &= \epsilon(r) & (108) \\ &+ \left(\frac{\mathbf{s}_1 \cdot \mathbf{L}_1}{2m_1^2} + \frac{\mathbf{s}_2 \cdot \mathbf{L}_2}{2m_2^2}\right)\frac{\epsilon'(r)}{r} & (109) \\ &+ \left(\frac{\mathbf{s}_1 \cdot \mathbf{L}_1}{m_1^2} - \frac{\mathbf{s}_2 \cdot \mathbf{L}_2}{m_2^2}\right)\frac{V_1'}{r} + \left(\frac{\mathbf{s}_2 \cdot \mathbf{L}_1}{m_1 m_2} - \frac{\mathbf{s}_1 \cdot \mathbf{L}_2}{m_1 m_2}\right)\frac{V_2'}{r} & (110) \\ &- \left(\frac{\mathbf{s}_1}{m_1} \cdot \frac{1}{T}\int_{-T/2}^{T/2} dz \langle \imath g \mathbf{B}(\mathbf{x}_1, z)\rangle_W / \langle 1 \rangle_W + (1 \to 2)\right) & (111) \end{aligned}$$

with

$$\begin{aligned} \left(\frac{\mathbf{s}_1 \cdot \mathbf{L}_1}{m_1^2} - \frac{\mathbf{s}_2 \cdot \mathbf{L}_2}{m_2^2}\right)\frac{V_1'}{r} &\equiv -\Big(\frac{\mathbf{s}_1}{2m_1^2} \cdot \frac{1}{T}\int\int_{-T/2}^{T/2} dz dz' \langle \mathbf{B}(\mathbf{x}_1, z)\mathbf{D}^2(\mathbf{x}_1, z')\rangle_W / \langle 1 \rangle_W \\ &\qquad + (1 \to 2)\Big) & (112) \\ (\mathbf{s}_2 \cdot \mathbf{L}_1 - \mathbf{s}_1 \cdot \mathbf{L}_2)\frac{V_2'}{r} &\equiv -\Big(\frac{\mathbf{s}_1}{2} \cdot \frac{1}{T}\int\int_{-T/2}^{T/2} dz dz' \langle \mathbf{B}(\mathbf{x}_1, z)\mathbf{D}^2(\mathbf{x}_2, z')\rangle_W / \langle 1 \rangle_W \\ &\qquad + (1 \leftrightarrow 2)\Big). & (113) \end{aligned}$$

The "large component only" interaction is precisely what is derived by authors Eichten and Feinberg in [2]. It's differences with the above V_{EF} potential are easily traced to three algebraic errors made in their derivation: 1) In their appendix there is an expression for S^{++}, (A3), given in terms of an integral whose evaluation is given in (A15). Direct substitution of (A15) into (A3) does not yield their equation (2.11) for S^{++}, but yields instead equation (53) above with spin terms on the rhs differing in algebraic sign. 2) There is a misrepresentation of the coordinate space momentum operator in going from equation (4.11a) to (4.11b) in the E.F. work. 3) There is a missed sign in the evaluation of an integral for $S^{--}(y, x)$ whose analogue in the evaluation of $S^{++}(x, y)$ is (A7b). Here this mistake amounts to using the integral result of (41) in the evaluation of $S^{--}(y, x)$ where the result of (76) is called for. These individual errors would have the following effects on the above V_{EF}: i)

error 1) would change the overall algebraic signs of lines (109) and(111). ii) error 2) would partially correct error 1) by again changing the algebraic sign of line (109). iii) error 3) would change the relative sign between terms in line (109) making the subscripted "2" term negative. The resulting interaction Hamiltonian would then be identical with the static plus spin-orbit interaction of reference [2].

The conspicuous lack of relative sign differences between subscripted "1" and "2" contributions in V_{EF} above makes its interpretation as a $Q\bar{Q}$ interaction Hamiltonian problematic. When the antiquark field operators in the beginning four point function (27) are properly charge conjugated (as they are for example in [5])this problem disappears. Specifically, the necessary change is

$$\begin{aligned} \psi(x_2) &\rightarrow \psi^c(x_2) & (114)\\ \Rightarrow S(x_2,y_2;A) &\rightarrow C^{-1}S(x_2,y_2;A)C = -\Big[S(y_2,x_2;-A^T)\Big]^T & (115)\end{aligned}$$

where the outer transposition in the last step is taken over both gauge and Dirac matrices. The effect on V_{EF} of (108) from the above charge conjugation made in (28) is simply to change the algebraic signs of the field insertions on antiquark lines of the Wilson loop while preserving the original path ordering

$$\begin{aligned} V_{EF,c} &= \epsilon(r) & (116)\\ &+ \left(\frac{\mathbf{s}_1\cdot\mathbf{L}_1}{2m_1^2} - \frac{\mathbf{s}_2\cdot\mathbf{L}_2}{2m_2^2}\right)\frac{\epsilon'(r)}{r} & (117)\\ &+ \left(\frac{\mathbf{s}_1\cdot\mathbf{L}_1}{m_1^2} - \frac{\mathbf{s}_2\cdot\mathbf{L}_2}{m_2^2}\right)\frac{V_1'}{r} + \left(\frac{\mathbf{s}_2\cdot\mathbf{L}_1}{m_1m_2} - \frac{\mathbf{s}_1\cdot\mathbf{L}_2}{m_1m_2}\right)\frac{V_2'}{r} & (118)\\ &- \left(\frac{\mathbf{s}_1}{m_1}\cdot\frac{1}{T}\int_{-T/2}^{T/2}dz\langle\imath g\mathbf{B}(\mathbf{x}_1,z)\rangle_W/\langle 1\rangle_W - (1\rightarrow 2)\right) & (119)\end{aligned}$$

with

$$\begin{aligned}\left(\frac{\mathbf{s}_1\cdot\mathbf{L}_1}{m_1^2} - \frac{\mathbf{s}_2\cdot\mathbf{L}_2}{m_2^2}\right)\frac{V_1'}{r} &\equiv -\Big(\frac{\mathbf{s}_1}{2m_1^2}\cdot\frac{1}{T}\int\int_{-T/2}^{T/2}dzdz'\langle\mathbf{B}(\mathbf{x}_1,z)\mathbf{D}^2(\mathbf{x}_1,z')\rangle_W/\langle 1\rangle_W \\ &\qquad -(1\rightarrow 2)\Big) & (120)\end{aligned}$$

$$(\mathbf{s}_2\cdot\mathbf{L}_1-\mathbf{s}_1\cdot\mathbf{L}_2)\,\frac{V_2'}{r} \;\equiv\; -\Big(\frac{\mathbf{s}_1}{2}\cdot\frac{1}{T}\int\int_{-T/2}^{T/2}dzdz'\langle\mathbf{B}(\mathbf{x}_1,z)\mathbf{D}^2(\mathbf{x}_2,z')\rangle_W/\langle 1\rangle_W -(1\leftrightarrow 2)\Big)\,. \tag{121}$$

where the "c" subscript indicates that antiquark fields have been charge conjugated. This is our V of equation(12).

To obtain the full V_I charge conjugated interaction likewise simply requires that we change the algebraic signs of the field insertions along the antiquark lines appearing in V_I above

$$V_{I,c} = \epsilon(r) \tag{122}$$

$$+\left(\frac{\mathbf{s}_1\cdot\mathbf{L}_1}{m_1^2}-\frac{\mathbf{s}_2\cdot\mathbf{L}_2}{m_2^2}\right)\frac{V_1'}{r}+\left(\frac{\mathbf{s}_2\cdot\mathbf{L}_1}{m_1m_2}-\frac{\mathbf{s}_1\cdot\mathbf{L}_2}{m_1m_2}\right)\frac{V_2'}{r} \tag{123}$$

$$-\left(\frac{\mathbf{s}_1}{m_1}\cdot\frac{1}{T}\int_{-T/2}^{T/2}dz\langle\imath g\mathbf{B}(\mathbf{x}_1,z)\rangle_W/\langle 1\rangle_W-(1\rightarrow 2)\right) \tag{124}$$

with

$$\left(\frac{\mathbf{s}_1\cdot\mathbf{L}_1}{m_1^2}-\frac{\mathbf{s}_2\cdot\mathbf{L}_2}{m_2^2}\right)\frac{V_1'}{r} \;\equiv\; -\Big(\frac{\mathbf{s}_1}{2m_1^2}\cdot\frac{1}{T}\int\int_{-T/2}^{T/2}dzdz'\langle\mathbf{B}(\mathbf{x}_1,z)\mathbf{D}^2(\mathbf{x}_1,z')\rangle_W/\langle 1\rangle_W -(1\rightarrow 2)\Big) \tag{125}$$

$$(\mathbf{s}_2\cdot\mathbf{L}_1-\mathbf{s}_1\cdot\mathbf{L}_2)\,\frac{V_2'}{r} \;\equiv\; -\Big(\frac{\mathbf{s}_1}{2}\cdot\frac{1}{T}\int\int_{-T/2}^{T/2}dzdz'\langle\mathbf{B}(\mathbf{x}_1,z)\mathbf{D}^2(\mathbf{x}_2,z')\rangle_W/\langle 1\rangle_W -(1\leftrightarrow 2)\Big)\,. \tag{126}$$

Interaction potential from the covariant 4-point function :

Here we derive the interaction potential from the covariant 4-point function whose antiquark fields are charge conjugated.

$$G_{cov} = \langle tr_\lambda S(x_1,y_1;A)P(y_1,y_2)C^{-1}S(y_2,x_2;A)CP(x_2,x_1)\rangle \tag{127}$$

$$\equiv -\tilde{G}e^{-i(m_1+m_2)T}\delta(\mathbf{x}_1-\mathbf{y}_1)\delta(\mathbf{x}_2-\mathbf{y}_2)\,. \tag{128}$$

The interaction is found from the large component of the diagonalized $\tilde{G}$ by

$$V_{cov,c} = -\frac{1}{T}\ln\tilde{G}_l \tag{129}$$

where $\tilde{G}_l$ is the large component. The diagonalization to $O(m^{-2})$ is performed via the Foldy-Wouthysen transformation $U = \exp(\imath s(\xi))$, with, $s(\xi) = \imath\gamma^0\boldsymbol{\gamma}\cdot\mathbf{D}(\xi)/2m$. It will suffice to reduce the off diagonal elements to $O(m^{-2})$, accomplished by a single transformation. For the quark propagator reduction, using the above results

$$\begin{aligned}
S^{++}(x,y) &= S_0^{++}(x,y) + \int d^4w S_0^{++}(x,w)\frac{1}{2m}(\boldsymbol{\gamma}\cdot\mathbf{D}(w))^2 S_0^{++}(w,y) && (130)\\
&\quad +O(m^{-2}) && (131)\\
S^{+-}(x,y) &= -\frac{1}{2m}S_0^{++}(x,y)\boldsymbol{\gamma}\cdot\mathbf{D}(y) + O(m^{-2}) && (132)\\
S^{-+}(x,y) &= -\frac{1}{2m}\boldsymbol{\gamma}\cdot\mathbf{D}(x)S_0^{++}(x,y) + O(m^{-2}) && (133)\\
S^{--}(x,y) &\sim O(m^{-2}) && (134)
\end{aligned}$$

we have

$$\begin{aligned}
S(x,y) \;\rightarrow\; & S'(x,y) = e^{iu(x)}S(x,y)e^{-iu(y)} && (135)\\
= \; & S(x,y) + i[u(x)(S^{++}(x,y) + S^{+-}(x,y) + S^{-+}(x,y)) && (136)\\
& \quad -(S^{++}(x,y) + S^{+-}(x,y) + S^{-+}(x,y))u(y) && (137)\\
& -\frac{1}{2}[u(x)(u(x)S^{++}(x,y) - S^{++}(x,y)u(y)) && (138)\\
& \quad -(u(x)S^{++}(x,y) - S^{++}(x,y)u(y))u(y)] + O(m^{-3}) && (139)\\
= \; & S^{++}(x,y) + \frac{1}{8m^2}[(\boldsymbol{\gamma}\cdot\mathbf{D}(x))^2 S_0^{++}(x,y) + S_0^{++}(x,y)(\boldsymbol{\gamma}\cdot\mathbf{D}(y))^2] \\
& +\frac{1}{4m^2}\int d^4w[S_0^{++}(x,w)(\boldsymbol{\gamma}\cdot\mathbf{D}(w))^2 S_0^{++}(w,y)\boldsymbol{\gamma}\cdot\mathbf{D}(y) && (140)\\
& \quad -\boldsymbol{\gamma}\cdot\mathbf{D}(x)S_0^{++}(x,w)(\boldsymbol{\gamma}\cdot\mathbf{D}(w))^2 S_0^{++}(w,y)] + O(m^{-3}) && (141)
\end{aligned}$$

The off diagonals are now of $O(m^{-2})$. Another transformation would further reduce the off diagonals to $O(m^{-3})$ while contributing nothing to the diagonal. The diagonalized quark large component is therefore given by $S^{++}(x,y)$ alone (with gauge

fields again set to zero at $\pm\frac{T}{2}$). The antiquark propagator diagonalization follows along the same lines, leading to the large antiquark component $C^{-1}S^{--}(y,x)C$. Then from (82) to (99) above we find

$$\tilde{G}_l = \tilde{I}_{l,c} \tag{142}$$

where the "c" subscript indicates that field insertions on the antiquark line of the Wilson loop in $\tilde{I}_l$ of (99) have undergone an algebraic sign change. This gives

$$\begin{aligned} V_{cov,c} &= V_{EF,c} &(143)\\ &= V &(144)\end{aligned}$$

where $V_{EF,c}$ is given in (116) and V in (12). I.e. the interaction obtained from the diagonalized covariant 4-point function is identical with the one obtained from the large component of the invariant interaction.

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